Sum of Sierpinski-Zygmund and Darboux Like Functions

Krzysztof Plotka*
Department of Mathematics, West Virginia University
Morgantown, WV 26506-6310, USA
kplotka@math.wvu.edu
and
Institute of Mathematics, Gdansk University
Wita Stwosza 57, 80-952 Gdansk, Poland

February 28, 2000

Abstract

For $\mathcal{F}_1, \mathcal{F}_2 \subseteq \mathbb{R}^{\mathbb{R}}$ we define $\text{Add}(\mathcal{F}_1, \mathcal{F}_2)$ as the smallest cardinality of a family $F \subseteq \mathbb{R}^{\mathbb{R}}$ for which there is no $g \in \mathcal{F}_1$ such that $g + F \subseteq \mathcal{F}_2$. The main goal of this note is to investigate the function Add in the case when one of the classes $\mathcal{F}_1, \mathcal{F}_2$ is the class SZ of *Sierpinski-Zygmund* functions. In particular, we show that *Martin's Axiom* (MA) implies $\text{Add}(\text{AC}, \text{SZ}) \geq \omega$ and $\text{Add}(\text{SZ}, \text{AC}) = \text{Add}(\text{SZ}, \text{D}) = \mathfrak{c}$, where AC and D denote the families of *almost continuous* and *Darboux* functions, respectively. As a corollary we obtain that the proposition: *every function from* $\mathbb{R}$ *into* $\mathbb{R}$ *can be represented as a sum of Sierpinski-Zygmund and almost continuous functions* is independent of ZFC axioms.

1 Introduction

The terminology is standard and follows [2]. The symbols $\mathbb{R}$ and $\mathbb{Q}$ stand for the sets of all real and all rational numbers, respectively. A basis of $\mathbb{R}$ as a linear space over $\mathbb{Q}$ is called *Hamel basis*. For $Y \subseteq \mathbb{R}$, the symbol $\text{Lin}_{\mathbb{Q}}(Y)$ stands for the smallest linear subspace of $\mathbb{R}$ over $\mathbb{Q}$ that contains Y. The cardinality of a set X we denote by $|X|$. In particular, $|\mathbb{R}|$ is denoted by $\mathfrak{c}$. Given a cardinal κ, we let $\text{cf}(\kappa)$ denote the cofinality of κ. We say that a cardinal κ is regular provided that $\text{cf}(\kappa) = \kappa$.

$\mathcal{B}$ and $\mathcal{M}$ stand for the families of all Borel and all meager subsets of $\mathbb{R}$, respectively. We say that a set $B \subseteq \mathbb{R}$ is a *Bernstein* set if both B and $\mathbb{R} \setminus B$

*This paper was written under supervision of K. Ciesielski. The author wishes to thank him for many helpful conversations.

intersect every perfect set. For a cardinal number κ, a set $A \subseteq \mathbb{R}$ is called κ-*dense* if $|A \cap I| \geq \kappa$ for every non-trivial interval I. For any planar set P, we denote its x-projection by $\mathrm{dom}(P)$.

We consider only real-valued functions. No distinction is made between a function and its graph. For any two partial real functions f, g we write $f+g$, $f-g$ for the sum and difference functions defined on $\mathrm{dom}(f) \cap \mathrm{dom}(g)$. The class of all functions from a set X into a set Y is denoted by Y^X. We write $f|A$ for the restriction of $f \in Y^X$ to the set $A \subseteq X$. For $B \subseteq \mathbb{R}^n$ its characteristic function is denoted by χ_B. If $f, g \in Y^X$, we denote the set $\{x \in X : f(x) = g(x)\}$ by $[f = g]$. For any function $g \in \mathbb{R}^X$ and any family of functions $F \subseteq \mathbb{R}^X$ we define $g + F = \{g + f : f \in F\}$.

The cardinal function $\mathrm{A}(F)$, for $F \subseteq \mathbb{R}^X$, is defined as the smallest cardinality of a family $F \subseteq \mathbb{R}^X$ for which there is no $g \in \mathbb{R}^X$ such that $g + F \subseteq F$. It was investigated for many different classes of real functions, see e.g. [5], [6], [13]. In this paper we generalize the function A by imposing some restrictions on the function g. Thus for $F_1, F_2 \subseteq \mathbb{R}^X$ we define

$$\mathrm{Add}(F_1, F_2) = \min\{|F| : F \subseteq \mathbb{R}^X \;\&\; \neg\exists g \in F_1\; g + F \subseteq F_2\} \cup \{(|\mathbb{R}^X|)^+\}.$$

Observe that $\mathrm{A}(F) = \mathrm{Add}(\mathbb{R}^X, F)$ for any set X, so the function Add is indeed a generalization of the function A. Notice also the following properties of the Add function.

Proposition 1 *Let $F_1 \subseteq F_2 \subseteq \mathbb{R}^X$ and $F \subseteq \mathbb{R}^X$.*

(1) $\mathrm{Add}(F_1, F) \leq \mathrm{Add}(F_2, F)$.

(2) $\mathrm{Add}(F, F_1) \leq \mathrm{Add}(F, F_2)$.

(3) $\mathrm{Add}(F_1, F_2) \geq 2$ *if and only if* $\mathbb{R}^X = F_2 - F_1$.

(4) *If* $\mathrm{Add}(F_1, F_2) \geq 2$ *then* $F_1 \cap F_2 \neq \emptyset$.

(5) $\mathrm{A}(F) = \mathrm{Add}(F, F) + 1$. *In particular, if* $\mathrm{A}(F) \geq \omega$ *then* $\mathrm{Add}(F, F) = \mathrm{A}(F)$.[1]

Proof. The properties (1)-(4) are obvious. We will prove (5). It is clear that $\mathrm{Add}(F, F) \leq \mathrm{A}(F)$. On the other hand, observe that $\mathrm{A}(F) \leq \mathrm{Add}(F, F) + 1$. To see the above let $F \subseteq \mathbb{R}^X$ be such that $|F| = \mathrm{Add}(F, F)$ and

$$\neg\exists\, g \in F\;\; g + F \subseteq F.$$

Then we have

$$\neg\exists\, g \in \mathbb{R}^X\;\; g + (F \cup \{0\}) \subseteq F,$$

where $0 \colon X \to \mathbb{R}$ is a function identically equal to zero.

[1] Very similar observation, in a little bit different context, was obtained independently by Francis Jordan [8, Proposition 1.3].

So the conclusion is obvious in the case $\mathrm{A}(F) \geq \omega$. Therefore we will concentrate on the case $\mathrm{A}(F) = k$ for some $k \in \omega$. Recall that the function A is bounded from the bottom by 1, thus $k \geq 1$. From the previous argument we imply that $\mathrm{Add}(F,F) \geq k - 1$. So we only need to justify that $\mathrm{Add}(F,F) \leq k - 1$.

Let $\{f_1, \ldots, f_k\}$ be a family witnessing $\mathrm{A}(F) = k$. Then the set $\{f_1 - f_k, \ldots, f_{k-1} - f_k\}$ witnesses $\mathrm{Add}(F,F) \leq k - 1$. Indeed, assume by contradiction, that we can find a function $f \in F$ such that $(f_i - f_k) + f \in F$ for every $i = 1, \ldots, k - 1$. Then the function $f - f_k$ shifts the set $\{f_1, \ldots, f_k\}$ into F. Contradiction. ■

Our main goal is to investigate the function Add in the case when one of the classes F_1, F_2 is the class of *Sierpinski-Zygmund* functions. Before we state the main result of the paper, let us recall the following definitions.

For $X \subseteq \mathbb{R}^n$ a function $f : X \to \mathbb{R}$ is:

- *additive* if $f(x + y) = f(x) + f(y)$ for all $x, y \in X$ such that $x + y \in X$;
- *almost continuous* (in sense of Stallings) if each open subset of $X \times \mathbb{R}$ containing the graph of f contains also graph of a continuous function from X to $\mathbb{R}$;
- *connectivity* if the graph of $f|Z$ is connected in $Z \times \mathbb{R}$ for any connected subset Z of X;
- *countably continuous* if it can be represented as a union of countably many continuous partial functions;
- *Darboux* if $f[K]$ is a connected subset of $\mathbb{R}$ (i.e., an interval) for every connected subset K of X;
- an *extendability* function provided there exists a connectivity function $F : X \times [0,1] \to \mathbb{R}$ such that $f(x) = F(x, 0)$ for every $x \in X$;
- *peripherally continuous* if for every $x \in X$ and for all pairs of open sets U and V containing x and $f(x)$, respectively, there exists an open subset W of U such that $x \in W$ and $f[\mathrm{bd}(W)] \subseteq V$;
- *Sierpinski-Zygmund* if for every set $Y \subseteq X$ of cardinality continuum $\mathfrak{c}$, $f|Y$ is discontinuous.

The classes of functions defined above are denoted by $\mathrm{AD}(X)$, $\mathrm{AC}(X)$, $\mathrm{Conn}(X)$, $\mathrm{CC}(X)$, $\mathrm{D}(X)$, $\mathrm{Ext}(X)$, $\mathrm{PC}(X)$, and $\mathrm{SZ}(X)$, respectively. The family of all continuous functions from X into $\mathbb{R}$ is denoted by $\mathrm{C}(X)$. We drop the index X in the case $X = \mathbb{R}$

set. (See [10].) It is also well-known that each continuous partial function can be extended to a continuous function defined on some G_δ set. (See [12].) Thus if $|[f = g]| < \mathfrak{c}$ for each continuous partial function g defined on some G_δ-set then f is Sierpinski-Zygmund. Recall also that each additive function $f \in \mathrm{AD}$ is linear over $\mathbb{Q}$, i.e., for all $p, q \in \mathbb{Q}$ and $x, y \in \mathbb{R}$ we have $f(px + qy) = pf(x) + qf(y)$.

The above classes are related in the following way (arrows $\rightarrow$ indicate proper inclusions.) (See [3] or [7].)

$$\mathrm{C} \longrightarrow \mathrm{Ext} \longrightarrow \mathrm{AC} \longrightarrow \mathrm{Conn} \longrightarrow \mathrm{D} \longrightarrow \mathrm{PC}$$

For functions from $\mathbb{R}$ into $\mathbb{R}$.

$$\mathrm{C}(\mathbb{R}^n) \longrightarrow \mathrm{Ext}(\mathbb{R}^n) = \mathrm{Conn}(\mathbb{R}^n) = \mathrm{PC}(\mathbb{R}^n) \longrightarrow \mathrm{AC}(\mathbb{R}^n) \cap \mathrm{D}(\mathbb{R}^n) \begin{array}{l} \nearrow \mathrm{AC}(\mathbb{R}^n) \\ \searrow \mathrm{D}(\mathbb{R}^n) \end{array}$$

For functions from $\mathbb{R}^n$ into $\mathbb{R}$ with $n \geq 2$.

The class of *Sierpinski-Zygmund* functions is independent of all the classes included in the above chart in the following sense. There is no inclusion between SZ and AC, Conn, D, or PC. SZ is disjoint with C and Ext. (See also comment below Corollary

The following remains an open problem. (See Fact 15.)

Problem 3 Does the equality $\mathrm{Add}(\mathrm{AC}, \mathrm{SZ}) = \omega$ hold in "ZFC + MA" (or in "ZFC + CH"?)

Let us make here some comments about the theorem. Parts (1) and (3) give only lower bound for $\mathrm{Add}(\mathrm{AC}, \mathrm{SZ})$. So one may wonder whether it is possible to give in ZFC any non-trivial upper bound for that number. However, in the model used to prove (3) it is possible to have $\mathfrak{c}^+ = 2^{\mathfrak{c}}$, so it cannot be proved in ZFC that $\mathrm{Add}(\mathrm{AC}, \mathrm{SZ}) < 2^{\mathfrak{c}}$. But it is unknown whether $\mathrm{Add}(\mathrm{AC}, \mathrm{SZ}) \leq \mathfrak{c}^+$ in ZFC. The next comment is about symmetry of Add. It is consistent that $A(\mathrm{SZ}) < 2^{\mathfrak{c}}$. (See [5].) Hence the part (4) implies that Add is not symmetric in general.

Next we give some corollaries of the main result. To state the first one, note that $-\mathrm{SZ} = \{-f : f \in \mathrm{SZ}\} = \mathrm{SZ}$. This observation, Proposition 1 and the part (2) of Theorem 2 immediately imply the following corollary.

Corollary 4 (MA) *Every function $f : \mathbb{R} \to \mathbb{R}$ can be represented as a sum of almost continuous and Sierpinski-Zygmund functions.*

Proof. For every n 2 if f 2 AC($\mathbb{R}^n$) \ SZ($\mathbb{R}^n$) then $fj\mathbb{R}^2$ 2 AC($\mathbb{R}^2$) \ SZ($\mathbb{R}^2$). (See [13].) Hence it is enough to prove the remark for $n = 2$. We construct the family fB_y : y 2 Rg of c-many blocking sets in $\mathbb{R}^3$ with pairwice8.C(

$f_{\alpha_B}|J \subseteq B$. From the definition of α_B and MA we see that $\bigcup_{\beta<\alpha_B}[f_\beta = q]$ is of first category as the union of less than $\mathfrak{c}$-many sets of first category. Recall that $F \in \mathcal{F}_A$. This implies that $(I \cap A)\setminus \bigcup_{\beta<\alpha_B}\bigcup_{f\in F}[(f_\beta - f) = q]$ is of second category for every nontrivial interval I. The above holds because otherwise we would have that $(K \cap A) \subseteq \bigcup_{\beta<\alpha_B}\bigcup_{f\in F}[(f_\beta - f) = q]$ for some $K \in \mathcal{B}\setminus\mathcal{M}$. Then for every $x \in (K\cap A)$ there are $\beta < \alpha_B$ and $f \in F$ such that $f_\beta(x) - f(x) = q(x)$. Define $h\colon (K\cap A) \to \mathbb{R}$ by $h(x) = f_\beta(x) - q(x) = f(x)$. It is easy to see that h is a subset of both $\bigcup_{\beta<\alpha_B}(f_\beta - q)$ and $\bigcup F$. In particular, it implies that $h \in \mathcal{C}^{<\mathfrak{c}}(K \cap A)$ which contradicts the assumption that $F \in \mathcal{F}_A$.

Hence $(J \cap A)\setminus \bigcup_{\beta<\alpha_B}(\bigcup_{f\in F}[(f_\beta - f) = q] \cup [f_\beta = q] \cup D_\beta)$ is of second category. Therefore $D_{\alpha_B} \cap J \neq \emptyset$. This implies $g' \cap B \supseteq g_{\alpha_B} \cap B \neq \emptyset$ (g_{α_B} and f_{α_B} coincide on $D_{\alpha_B} \cap J$).

(2) Let $g''\colon A \setminus \mathrm{dom}(g') \to \mathbb{R}$ be a Sierpinski-Zygmund function such that $g'' + F \subseteq \mathrm{SZ}_{\mathrm{part}}$. Such a function exists because $|F| < \mathrm{A}(\mathrm{SZ})$. We define $g = g' \cup g''$. We see that $g \in \mathrm{SZ}(A)$, any extension of g onto $\mathbb{R}$ is in AC, and $g + F \subseteq \mathrm{SZ}(A)$. ∎

Lemma 13 (MA) *Let $\{f_i\}_1^n \subseteq \mathbb{R}^{\mathbb{R}}$, $n = 1, 2, \ldots$. There exists $\{f_i'\}_1^n \in \mathcal{F}_A$ such that $f_i|A_i \in \mathcal{C}^{<\mathfrak{c}}(A_i)$, where $A_i = [f_i \neq f_i']$.*

Proof. The proof is by induction on number n of functions.

Assume that the lemma is true for every $\{g_i\}_1^{n-1} \subseteq \mathbb{R}^{\mathbb{R}}$, $n \geq 1$. Let us fix $\{f_i\}_1^n \subseteq \mathbb{R}^{\mathbb{R}}$. We will construct a family $\{f_i'\}_1^n \in \mathcal{F}_A$ such that $f_i|[f_i \neq f_i'] \in \mathcal{C}^{<\mathfrak{c}}([f_i \neq f_i'])$ for all $i \leq n$.

We start with showing that the following claim holds for all $f, h, h' \in \mathbb{R}^{\mathbb{R}}$.

$$\text{If } f|[f \neq h] \in \mathcal{C}^{<\mathfrak{c}}_{\mathrm{part}} \text{ and } h|[h \neq h'] \in \mathcal{C}^{<\mathfrak{c}}_{\mathrm{part}} \text{ then } f|[f \neq h'] \in \mathcal{C}^{<\mathfrak{c}}_{\mathrm{part}}.$$

This is so because we have that $[f \neq h'] \subseteq [f \neq h] \cup [h \neq h']$ and consequently

$$f|[f \neq h'] \subseteq f|([f \neq h] \cup [h \neq h']) = f|[f \neq h] \cup f|([h \neq h'] \setminus [f \neq h])$$

$$\subseteq f|[f \neq h] \cup h|[h \neq h'].$$

This completes the proof of the claim.

Now observe that, by the inductive assumption, there exists $\{h_i\}_2^n \in \mathcal{F}_A$ such that $f_i|[f_i \neq h_i] \in \mathcal{C}^{<\mathfrak{c}}_{\mathrm{part}}$

f91t

There exists a maximal element $A_{\max}$ in $B_{f_1;:::;f_n}$ with respect to the relation * de ned by

$$X_1 \quad ^* X_2; \text{ if } X_1 \, n \, X_2 \text{ is of \quad rst category.}$$

in $\mathbb{R}^2$, we obtain that $\bigcup_{n<\omega} f_n|A_n$ is also meager as a union of countably many meager sets. We conclude from this that there exists a meager horizontal section of $\bigcup_{n<\omega} f_n|A_n$. Therefore the set $\bigcup F \setminus \bigcup_{n<\omega} f_n|A_n$ contains a constant function defined on comeager Borel set. ■

Using very similar technique as the above we can prove

Fact 15 (CH) Either $\mathrm{Add}(\mathrm{AC}, \mathrm{SZ}) = \omega$ or $\mathrm{Add}(\mathrm{AC}, \mathrm{SZ}) > \mathfrak{c}$.

Proof. Let us assume that $F = \{f_\xi : \xi < \mathfrak{c}\} \subseteq \mathbb{R}^\mathbb{R}$ witnesses $\mathrm{Add}(\mathrm{AC}, \mathrm{SZ}) \leq \mathfrak{c}$. For every $n < \omega$, define a function f_n^* as an extention of $\bigcup_{\xi<\mathfrak{c}} f_\xi|M_\xi^n$ onto $\mathbb{R}$, where $\{M_\xi^n : n < \omega, \xi < \mathfrak{c}\}$ is an Ulam matrix. We claim that $\{f_n^* : n < \omega\}$ witnesses $\mathrm{Add}(\mathrm{AC}, \mathrm{SZ}) \leq \omega$. To see this fix an $h \in \mathrm{AC}$. By our assumption about F, there exists an $\xi_0 < \mathfrak{c}$ such that $h + f_{\xi_0} \notin \mathrm{SZ}$. That means $h + f_{\xi_0}$ is continuous on a set X of cardinality continuum. Since $\mathbb{R} \setminus \bigcup_{n<\omega} M_{\xi_0}^n$ is countable we obtain that $|X \cap M_{\xi_0}^m| = \mathfrak{c}$ for some $m < \omega$. Hence $h + f_m^*$ is continuous on a set of cardinality continuum which means that $h + f_m^* \notin \mathrm{SZ}$. ■

Proof of $\mathrm{Add}(\mathrm{AC}, \mathrm{SZ}) \geq \omega$ (under MA).

We begin by fixing $F = \{f_1, \dots, f_n\} \subseteq \mathbb{R}^\mathbb{R}$. Let $F' = \{f_1', \dots, f_n'\} \in \mathcal{F}_\mathbb{R}$ be a corresponding family given by Lemma 13 for $A = \mathbb{R}$. Based on Lemma 12, we can find a $g \in \mathrm{AC} \cap \mathrm{SZ}$ such that $g + F' \subseteq \mathrm{SZ}$. Since $f_i|[f_i' \neq f_i] \in \mathcal{C}_{\mathrm{part}}^{<\mathfrak{c}}$ and $g \in \mathrm{SZ}$, we obtain that $g + f_i \in \mathrm{SZ}$ (for $i = 1, 2, \dots, n$.) ■

In order to prove part (2) of Theorem 2 we need to state one more lemma.

Lemma 16 $\mathrm{Add}(\mathrm{SZ}, \mathrm{D}) \geq 2^{<\mathfrak{c}}$.

Proof.

Since $f'|[f' \neq f_\alpha] \in \mathrm{C}^{<\mathfrak{c}}_{part}$ and $\mathrm{SZ}(X) + \mathrm{C}^{<\mathfrak{c}}(X) = \mathrm{SZ}(X)$ for every $X \subseteq \mathbb{R}$, we conclude that $g_\alpha + f_\alpha \in \mathrm{SZ}_{part}$, $\alpha < \beta$. Put $h = \bigcup_{\alpha<\beta}(g_\alpha + f_\alpha)$. Since Martin's Axiom implies the regularity of $\mathfrak{c}$ we obtain that $h \in \mathrm{SZ}$. Clearly, $h + F \subseteq \mathrm{AC}$. ■

As the final remark let us notice that parts (1) and (2) of the main result as well as Lemmas 12 and 13 could be proved under weaker assumptions. The proofs require only two consequences of Martin's Axiom: $\mathfrak{c} = \mathfrak{c}^{<\mathfrak{c}}$ (this implies regularity of $\mathfrak{c}$); the union of less than $\mathfrak{c}$-many meager sets is meager.

3 Proof of Theorem 2 (3)

We will show that the existence of $\mathfrak{c}$-additive ω_1-saturated ideal $\mathcal{J}$ in $\mathcal{P}(\mathbb{R})$ containing $\mathcal{M}$ implies $\mathrm{Add}(\mathrm{AC}, \mathrm{SZ}) > \mathfrak{c}$. It is known that the existence of such an ideal is equiconsistent with "ZFC + $\exists$ measurable cardinal."[2] (See [9].)

First notice that we may assume that $\mathcal{J} \cap \mathcal{B} = \mathcal{M}$. To see this suppose that there exists a Borel set B of second category in $\mathcal{J}$. B is residual in some open interval I. Then $I \in \mathcal{J}$ because $I \setminus B$ is meager and $I = (B \cap I) \cup (I \setminus B)$. Now, let U be a maximal open set belonging to $\mathcal{J}$. Such a set exists because the union of all open sets from $\mathcal{J}$ can be represented as a union of countable many such sets. We have that $\mathbb{R} \setminus U$ contains a nonempty open interval I_0. Otherwise it would be nowhere-dense and then $\mathbb{R} = U \cup (\mathbb{R} \setminus U) \in \mathcal{J}$. Now, any homeomorphism between I_0 and $\mathbb{R}$ induces the desired ideal on $\mathbb{R}$.

The schema of the proof is similar to the idea of combining Lemmas 12 and 13 in the proof of $\mathrm{Add}(\mathrm{AC}, \mathrm{SZ})$ [illegible] $\to$. First step is to show that

(∗) for each $f: \mathbb{R} \to \mathbb{R}$ there exists an $f^{\mathcal{J}} \in \mathbb{R}^{\mathbb{R}}$ such that $f|[f \neq f^{\mathcal{J}}] \in \mathrm{CC}_{part}$ and $f^{\mathcal{J}}|X \notin \mathrm{CC}(X)$ for every $X \notin \mathcal{J}$.

To see this fix an $f \in \mathbb{R}^{\mathbb{R}}$. We claim that there exists a set Y such that $f|Y \in \mathrm{CC}(Y)$ and $Y' \subseteq^{\mathcal{J}} Y$ for all Y' satisfying $f|Y' \in \mathrm{CC}(Y')$, where $\subseteq^{\mathcal{J}}$ is defined by

$$Z_1 \subseteq^{\mathcal{J}} Z_2, \text{ if } Z_1 \setminus Z_2 \in \mathcal{J}.$$

If the claim did not hold then we could easily construct a strictly increasing

In the next step we fix a family $\mathcal{F}$ of real functions of cardinality $\mathfrak{c}$. Let $\mathcal{F} = \{h : \ < \mathfrak{c}\}$ be an enumeration of $\mathcal{F}$ and $\langle f : \ < \mathfrak{c}\rangle$ be a sequence of all continuous functions defined on G subsets of $\mathbb{R}$. Based on the previous reasoning we may assume that $h|X \notin CC(X)$ for every $X \notin \mathcal{J}$ and $< \mathfrak{c}$. Notice that if $; < \mathfrak{c}$ and $f|X \subseteq \bigcup_{; <} (f \cap h)$ then $X \in \mathcal{J}$. This is so since $X \subseteq \bigcup_{; <} [f = f \cap h]$ and every set $[f = f \cap h] = [h = f \cap f] \in \mathcal{J}$. Consequently, the set $\mathrm{dom}(f \setminus \bigcup_{; <} (f \cap h))$ does not belong to $\mathcal{J}$ provided $\mathrm{dom}(f) \notin \mathcal{J}$.

Now we construct a sequence $\langle g : \ < \mathfrak{c}\rangle$ of partial functions such that

g is a countable dense subset of $f \setminus \bigcup_{; <} ((f \cap h) \cup f \cup L($

Proof. Let $\langle x_\alpha : \alpha < \mathfrak{c}\rangle$ and $\langle f_\alpha : \alpha < \mathfrak{c}\rangle$ be the sequences of all real numbers and all continuous functions defined on a G_δ subset of $\mathbb{R}$, respectively. We will define the set X by defining its vertical sections by transfinite induction. For every $\alpha < \mathfrak{c}$ we put

$$X_{x_\alpha} = \mathbb{R} \setminus \{f_\beta(x_\alpha) : \beta < \alpha\}.$$

Put $X = \bigcup_{\alpha<\mathfrak{c}} \{x_\alpha\} \times X_{x_\alpha}$. It is obvious that X has the required properties. ■

Corollary 20 *There exists a family $\{Q_x \subseteq \mathbb{R} : x \in \mathbb{R}\}$ of pairwise disjoint countable dense sets such that $\bigcup_{x\in\mathbb{R}} \{x\} \times Q_x$ is an SZ-set.*

The next lemma is proved in [6].

Lemma 21 [6, Lemma 2.2] *If $B \subseteq \mathbb{R}$ has cardinality $\mathfrak{c}$ and $H \subseteq \mathbb{Q}^B$ is such that $|H| < 2^{\mathfrak{c}}$ then there is a $g \in \mathbb{Q}^B$ such that $h \cap g \neq \emptyset$ for every $h \in H$.*

We give more general version of this lemma.

Lemma 22 *If $B \subseteq \mathbb{R}$ has cardinality $\mathfrak{c}$ and $H \subseteq \prod_{x\in B} Q_x$ is such that $|H| < 2^{\mathfrak{c}}$ then there is a $g \in \prod_{x\in B} Q_x$ such that $h \cap g \neq \emptyset$ for every $h \in H$.*

is such H|<2(2)]TJ/F5827(44)-28(ectiv)23(ely)8v-11.9d1 9.2 n there i71969833(a)]TJ/F11 9.963 Tf 66.5 0

Proof. For every $x \in B$ let $f_x : Q_x \to \mathbb{Q}$ be a bijection. Now, for each $h \in H$ we define h' as follows

2h'(2)(x)]TJ/F58=F11 9.963 Tf 10.79 0(2(25)]TJ/F58 9.963 Tf 6.123 1.49539eryhj

Now, let $g \in \prod_{x\in\mathbb{R}} Q_x$ be a common extension of all functions $g_{\langle I;p;m\rangle}$. Corollary 20 implies that g is of Sierpinski-Zygmund type. The function g has also the following property. For every $\langle I;p;m\rangle \in \mathcal{G}$ and every $f \in \mathcal{F}$ there exists $x \in B_{\langle I;p;m\rangle} \subseteq I$ such that

$$|p - (f(x) + g(x))| < \frac{1}{m}.$$

So, each function $f + g$, for $f \in \mathcal{F}$, is dense in $\mathbb{R}^2$. Thus $f + g \in \mathrm{PC}$. ∎

5 Proofs of Theorems 9 and 10

In this section we present proofs of Theorems 9 and 10. Before we do this, let us recall some definitions and cite some theorems. Let $h \in \mathrm{Ext}$. We say that a set $G \subseteq \mathbb{R}$ is *h-negligible* provided $f \in \mathrm{Ext}$ for every function $f\colon \mathbb{R} \to \mathbb{R}$ for which $f = h$ on a set $\mathbb{R} \setminus G$. For a cardinal number $\kappa \le \mathfrak{c}$, a function $f\colon \mathbb{R} \to \mathbb{R}$ is called *κ strongly Darboux* if $f^{-1}(y)$ is κ-dense. If $\kappa = \omega$ then we simply say that f is strongly Darboux. We denote the family of all κ strongly Darboux functions by $\mathrm{D}(\kappa)$. It is obvious from the definition that

$$\mathrm{D}(\lambda) \subseteq \mathrm{D}(\kappa) \text{ for all cardinals } \kappa \le \lambda \le \mathfrak{c}.$$

We also introduce the family $\mathrm{D}(\mathrm{P})$ of *perfectly Darboux* functions as the class of all functions $f\colon \mathbb{R} \to \mathbb{R}$ such that $Q \cap f^{-1}(y) \neq \emptyset$ for every perfect set $Q \subseteq \mathbb{R}$ and $y \in \mathbb{R}$. In other words, a function f is perfectly Darboux if for every $y \in \mathbb{R}$ $f^{-1}(y)$ is a Bernstein set. Notice that $\mathrm{D}(\mathrm{P}) \subseteq \mathrm{D}(\kappa)$ for every $\kappa \le \mathfrak{c}$.

The following theorem is proved in [4].

Theorem 23. *$A(\mathrm{AC}) = A(\mathrm{D}) = A(\mathrm{D}(\omega_1))$.*

A little modification of the proof of the above theorem gives the following lemma.

Lemma 24 *Let $\mathcal{F} \in \{\mathrm{AD}, \mathrm{Ext}\}$. Then $\mathrm{Add}(\mathcal{F}, \mathrm{AC}) = \mathrm{Add}(\mathcal{F}, \mathrm{D})$.*

The proof of Lemma 24 requires the use of the following lemma and proposition.

Lemma 25 *Let X be any set of cardinality continuum and $F \subseteq \mathbb{R}^X$ satisfies the condition $|F| < A(\mathrm{D})$. There exists a $g\colon X \to \mathbb{R}$ such that $(g + f)^{-1}(y) \neq \emptyset$ for each $y \in \mathbb{R}$.*

Proof. Let $b\colon \mathbb{R} \to X$ be a bijection. By Theorem 23 and monotonicity of A we have that $A(\mathrm{D}) = A(\mathrm{D}(\omega))$. Hence we can find a $g'\colon \mathbb{R} \to \mathbb{R}$ satisfying the property that $g' + (f \circ b) \in \mathrm{D}(\omega)$ for each $f \in F$. Put $g = g' \circ b^{-1}$. Clearly, g is the desired function. ∎

Proposition 26 $A(D) = A(D(P))$.

Proof. Fix a family $F \subseteq \mathbb{R}^{\mathbb{R}}$ of cardinality less than $A(D)$. Next, let $\{B_\xi : \xi < \mathfrak{c}\}$ and $\{P_\xi : \xi < \mathfrak{c}\}$ be a family of pairwise disjoint Bernstein sets and an enumeration of all perfect subsets of $\mathbb{R}$, respectively. We define the sequence $\langle A_\xi : \xi < \mathfrak{c}\rangle$ by $A_\xi = B_\xi \cap P_\xi$. Obviously the sets A_ξ are pairwise disjoint and each one of them has cardinality $\mathfrak{c}$. Applying Lemma 25 for every $\xi < \mathfrak{c}$ separately, we get a sequence of functions $\langle g_\xi : A_\xi \to \mathbb{R} \mid \xi < \mathfrak{c}\rangle$ such that for every $\xi < \mathfrak{c}$ the following holds

$$\forall f \in F \ \forall y \in \mathbb{R} \ (g_\xi + f)^{-1}(y) \neq \emptyset.$$

Now, if $g \in \mathbb{R}^{\mathbb{R}}$ is any extension of $\bigcup_{\xi<\mathfrak{c}} g_\xi$ onto $\mathbb{R}$ then $g + F \subseteq D(P)$. ■

Proof of Lemma 24.

First we show that

$(*)$ $\mathrm{Add}(F, F_0) > \mathfrak{c}$ for $F_0 \in \{AC, D(\omega_1)\}$.

Let us fix a family $F \subseteq \mathbb{R}^{\mathbb{R}}$ with cardinality $\mathfrak{c}$. To prove the case $F = AD$ consider a $\mathfrak{c}$-dense Hamel basis H. There exists a partition $\{B_f : f \in F\}$ of H into $\mathfrak{c}$-dense sets. Since the projection of every blocking set in $\mathbb{R}^2$ contains an interval, we can find, for every $f \in F$, a partial function $g_f : B_f \to \mathbb{R}$ such that $g_f + f$ intersects every blocking set in at least ω_1 points. Thus every extension of $g_f + f$ onto $\mathbb{R}$ is almost continuous and ω_1 strongly Darboux. If $g \in \mathbb{R}^{\mathbb{R}}$ is any function containing $\bigcup_{f\in F} g_f$ then $g + F \subseteq AC \cap D(\omega_1)$. In particular, we can choose g to be an additive function. Hence $\mathrm{Add}(AD, F_0) > \mathfrak{c}$ for $F_0 \in \{AC, D(\omega_1)\}$.

Now consider the case $F = \mathrm{Ext}$. If $F_0 = AC$ then we have the inequality $\mathrm{Add}(\mathrm{Ext}, AC) \geq \mathrm{Add}(\mathrm{Ext}, \mathrm{Ext}) = A(\mathrm{Ext}) = \mathfrak{c}^+ > \mathfrak{c}$ which follows from Proposition 1 (2)&(5). Now, let us focus on the case $F_0 = D($

Consider a family $G \quad \mathbb{R}^{\mathbb{R}}$ of cardinality witnessing $= \mathrm{Add}(\mathcal{F}; \mathrm{D}(\omega_1))$. We define a new family $G^* = \{h \in \mathbb{R}^{\mathbb{R}} : \exists g \in G\ h =^* g\}$, where $h =^* f$ if and only if $|\{x : h(x) \neq f(x)\}| \quad \omega$. Notice here that $|G^*| = \quad$. This is so because $\quad > \mathfrak{c}$ and for every $f \in \mathbb{R}^{\mathbb{R}}$ the set $\{h \in \mathbb{R}^{\mathbb{R}} : h =^* f\}$ has cardinality $\mathfrak{c}$. We claim that G^* witnesses $\quad \mathrm{Add}(\mathcal{F}; \mathrm{D})$. Indeed, let $f \in \mathcal{F}$. Then, by the choice of G, there exists a $g \in G$ satisfying the following $f + g \notin \mathrm{D}(\omega_1)$. This implies the existence of a non-trivial closed interval I and $y \in \mathbb{R}$ for which $|I \cap (f+g)^{-1}(y)| \quad \omega$. By modification of g on a countable set, we get a function $g^* \in G^*$ with the property that $(f + g^*)[I] \cap (\ \infty; y) \neq \emptyset \neq (f + g^*)[I] \cap (y; \infty)$ and $y \notin (f + g^*)[I]$. Therefore $(f + g^*) \notin \mathrm{D}$. This ends the proof of the equality $\mathrm{Add}(\mathcal{F}; \mathrm{D}) = \mathrm{Add}(\mathcal{F}; \mathrm{D}(\omega_1))$.

What remains to show is that $\mathrm{Add}(\mathcal{F}; \mathrm{AC}) = \mathrm{Add}(\mathcal{F}; \mathrm{D}(\omega_1))$. The inequality $\mathrm{Add}(\mathcal{F}; \mathrm{AC}) \quad \mathrm{Add}(\mathcal{F}; \mathrm{D}) = \mathrm{Add}(\mathcal{F}; \mathrm{D}(\omega_1$

Fix a family $F \subseteq \mathbb{R}^\mathbb{R}$ of cardinality less than $2^\mathfrak{c}$. Now, a small modification in the proof of the equality $\mathrm{Add}(\mathrm{SZ}, \mathrm{PC}) = 2^\mathfrak{c}$ in Section 4 (the sets $B_{\langle I,p,m\rangle}$ can be chosen to be subsets of $\mathbb{R} \setminus Q$), gives us a function $g\colon \mathbb{R} \to \mathbb{R}$ which shifts F into PC and which agrees with f on the set containing Q. In particular, g is an extendable function.

(iv) The last part of Theorem 9 is proved by the following inequality

$$A(\mathrm{D}) = A(\mathrm{AC}) = \mathrm{Add}(\mathrm{AC}, \mathrm{AC}) \le \mathrm{Add}(F_1, F_2) \le \mathrm{Add}(\mathrm{D}, \mathrm{D}) = A(\mathrm{D}).$$

■

Proof of Theorem 10.

(i) To prove the first part of Theorem 10 we need one more lemma.

Lemma 27 $\mathrm{Add}(\mathrm{AD}, \mathrm{D}) \ge A(\mathrm{D}(\mathrm{P}))$. *In particular,* $\mathrm{Add}(\mathrm{AD}, \mathrm{D}) = A(\mathrm{D})$.

Proof. Let $P \subseteq \mathbb{R}$ be a perfect set with the property that $P \cup \{1\}$ is linearly independent over $\mathbb{Q}$. Observe that for every $p, q \in \mathbb{Q}$, $p \notin \{0, 1\}$ we have $(pP + q) \cap P = \emptyset$. Now, consider a countable partition $\{P_n : n < \omega\}$ of P into perfect sets. Using this partition and the above observation we can easily construct a family $\{P_n^\star : n < \omega\}$ of disjoint perfect sets such that $\bigcup_{n<\omega} P_n^\star$ is independent over $\mathbb{Q}$ and for every nontrivial interval $I \subseteq \mathbb{R}$ there is an $m < \omega$ such that $P_m^\star \subseteq I$. Note that $\bigcup_{n<\omega} P_n^\star$ is a $\mathfrak{c}$-dense meager F_σ-set.

To prove the inequality $\mathrm{Add}(\mathrm{AD}, \mathrm{D}) \ge A(\mathrm{D}(\mathrm{P}))$ let us fix a family $F \subseteq \mathbb{R}^\mathbb{R}$ such that $|F| < A(\mathrm{D}(\mathrm{P}))$. There exists a function $g \in \mathbb{R}^\mathbb{R}$ satisfying the property $g + F \subseteq \mathrm{D}(\mathrm{P})$. We claim that if $g^\star\colon \mathbb{R} \to \mathbb{R}$ is any additive extension of $g|\bigcup_{n<\omega} P_n^\star$ then $g^\star + F \subseteq \mathrm{D}$. More precisely, for every $f \in F$, $g^\star + f$ is strongly Darboux. To see this pick any $f \in F$, $y \in \mathbb{R}$, and any interval I. There exists $m < \omega$ such that $P_m^\star$ is contained in I. Furthermore, we can find $x \in P_m^\star \subseteq I$ for which $g^\star(x) + f(x) = g(x) + f(x) = y$. This shows that $g^\star + f$ is strongly Darboux.

The second statement in the lemma is proved by the obvious inequality $A(\mathrm{D}) \ge \mathrm{Add}(\mathrm{AD}, \mathrm{D}) \ge A(\mathrm{D}(\mathrm{P}))$ and Proposition 26. ■

Now, (i) follows from Lemmas 24, 27, and Proposition 1 (1).

(ii) Since $\mathrm{Add}(\mathrm{AD}, \mathrm{Ext}) \le A(\mathrm{Ext}) = \mathfrak{c}^+$, it suffices to show the inequality $\mathrm{Add}(\mathrm{AD}, \mathrm{Ext}) \ge \mathfrak{c}^+$. So for every $F = \{f_\xi : \xi < \mathfrak{c}\} \subseteq \mathbb{R}^\mathbb{R}$ we need to find a $g \in \mathrm{AD}$ such that $g + F \subseteq \mathrm{Ext}$.

Let $\langle D_\xi : \xi < \mathfrak{c}\rangle$ be a sequence of pairwise disjoint $\mathfrak{c}$-dense meager F_σ sets such that $\bigcup_{\xi<\mathfrak{c}} D_\xi$ is linearly independent over $\mathbb{Q}$. Such a sequence can be constructed in a similar way as the $\mathfrak{c}$-dense meager F_σ-set in the proof of Lemma 27. Now, by [3, Proposition 4.3], for every $\xi < \mathfrak{c}$ we can find $h_\xi \in \mathrm{Ext}$ such that $\mathbb{R} \setminus D_\xi$ is h_ξ-negligible. We define g as an additive extension of $\bigcup_{\xi<\mathfrak{c}} (h_\xi - f_\xi)|D_\xi$.

To see that $g + f_\xi \in \mathrm{Ext}$ for every ξ, observe that $g + f_\xi = h_\xi$ on D_ξ. But the set $\mathbb{R} \setminus D_\xi$ is h_ξ-negligible. So each $g + f_\xi$ is extendable.

(iii) The prove of this part is similar to the prove of Theorem 2 (4). Fix a Hamel basis H which is a Bernstein set. By choosing the sets $B_{\langle I,p,m\rangle}$ to

[7] R.G. Gibson, T. Natkaniec, *Darboux-like functions*