

The ideal of Sierpiński-Zygmund sets on the plane

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Abstract

We say that a set $X \subseteq \mathbb{R}^2$ is *Sierpiński-Zygmund* (shortly *SZ-set*) if it does not contain a partial continuous function of cardinality continuum c . We observe that the family of all such sets is $\text{cf}(c)$ -additive ideal. Some examples of such sets are given. We also consider *SZ-shiftable sets*, that is, sets $X \subseteq \mathbb{R}^2$ for which there exists a function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f + X$ is an SZ-set. Some results are proved about SZ-shiftable sets. In particular, we show that the union of two SZ-shiftable sets does not have to be SZ-shiftable.

The terminology is standard and follows [2]. The symbol \mathbb{R} stands for the set of all real numbers. The cardinality of a set X we denote by $|X|$. In particular, $|\mathbb{R}|$ is denoted by c . Given a cardinal κ , we let $\text{cf}(\kappa)$ denote the cofinality of κ . We say that a cardinal κ is regular provided that $\text{cf}(\kappa) = \kappa$.

A set $M \subseteq \mathbb{R}^n$ is called *Marczewski measurable* if every perfect set P has a perfect subset Q such that $Q \cap M = \emptyset$ or $Q \cap M = P$. If every perfect set P has a perfect subset Q such that $Q \cap M = \emptyset$, then M is called *Marczewski null*.

We consider only real-valued functions unless stated otherwise. No distinction is made between a function and its graph. For any planar set Y , we denote its x -projection by $\text{dom}(Y)$. For any two partial real functions f, g we write $f + g, f - g$

$g \in \mathbb{R}^X$, any family of functions $F \subseteq \mathbb{R}^X$, and any set $A \subseteq X \times \mathbb{R}$ we define $g + F = \{g + f : f \in F\}$ and $g + A = \{x, g(x) + y : x, y \in A\}$. The image and preimage of a set B under the function h are denoted by $h[B]$ and $h^{-1}[B]$, respectively.

Let us recall that a function $f: \mathbb{R} \rightarrow \mathbb{R}$ is *Sierpiński-Zygmund* ($f \in \text{SZ}$) if for every set $X \subseteq \mathbb{R}$ of cardinality continuum c , $f|_X$ is discontinuous. This definition is generalized onto subsets of \mathbb{R}^2 . (See [8].)

Definition 1 A set $X \subseteq \mathbb{R}^2$ is called *Sierpiński-Zygmund set* (shortly *SZ-set*), if for every partial real continuous function f we have $|f \upharpoonright X| < c$.

We denote the family of all SZ-sets by \mathcal{J}_{SZ} . Since every Sierpiński-Zygmund function is also an SZ-set we have that the family \mathcal{J}_{SZ} is not empty.

The next fact follows directly from the definition.

Fact 2 \mathcal{J}_{SZ} is a $\text{cf}(c)$ -additive ideal.

PROOF. It is obvious that \mathcal{J}_{SZ} is closed under the operation of taking subsets. We will show that szsets is $\text{cf}(c)$ -additive.

Take a $\kappa < \text{cf}(c)$. Let $\{X_\alpha : \alpha < \kappa\} \subseteq \mathcal{J}_{\text{SZ}}$ and $f \upharpoonright \bigcup_{\alpha < \kappa} X_\alpha$ be a partial continuous function. Since X_α is SZ-set, we have that $|f \upharpoonright X_\alpha| < c$ for each $\alpha < \kappa$. Consequently, $|f \upharpoonright \bigcup_{\alpha < \kappa} X_\alpha| = |\bigcup_{\alpha < \kappa} (f \upharpoonright X_\alpha)| < c$. ■

The question that one could ask here is how "big" an SZ-set can be. An example of the SZ-set that can be considered "big" in some sense is given in [8].

Lemma 3 [8, Lemma 19] *There exists an SZ-set $X \subseteq \mathbb{R}^2$ such that $|\mathbb{R} \setminus X_x| < c$ for every*

We claim that there exists an $A \subseteq \mathbb{R}^1$ such that $|\bigcup_{y \in A} X^y| < c$. The following two cases are possible.

Case 1. There exists a $\kappa < c$ such that $Z = \{y: |X^y| = \kappa\}$ is uncountable.

Then we choose $A \subseteq Z$. Obviously, $|\bigcup_{y \in A} X^y| = \kappa < c$.

Case 2. $|Z| < \kappa$ for every cardinal $\kappa < c$.

Put $Z = \{|X^y|: y \in \mathbb{R}\}$ and observe that $\mathbb{R} = \bigcup_{\alpha \in Z} Z_\alpha$. It follows from (*) that if $\alpha \in Z$ then $\alpha < c$. Consequently, since the union of less than continuum many countable sets has size less than continuum, we conclude that $|Z| = c$. Let α be the α -st element of Z . We define $A = \{y: |X^y| = \alpha\}$.

any more. However, if $h: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ is a homeomorphism preserving vertical lines then $h[X]$ is an SZ-set for every $X \in \mathcal{J}_{SZ}$.

Fact 6 Let $h: \mathbb{R}^2 \rightarrow \mathbb{R}^2$ be an homeomorphism such that $h[L]$ is a vertical line for every vertical line L . Then $h\{\mathcal{J}_{SZ}\} = \{h[X]: X \in \mathcal{J}_{SZ}\} = \mathcal{J}_{SZ}$.

PROOF. First we show the inclusion $h\{\mathcal{J}_{SZ}\} \subseteq \mathcal{J}_{SZ}$. It is easy to see that if $f: A \rightarrow \mathbb{R}$ is a partial continuous function then $h^{-1}[f]: A \rightarrow \mathbb{R}$ is also continuous. This implies that for every $X \in \mathcal{J}_{SZ}$, $h[X]$ is also in \mathcal{J}_{SZ} .

Now to show the other inclusion, let us fix a $Y \in \mathcal{J}_{SZ}$. Note that h^{-1} also preserves all vertical lines. Thus, from the first part of the proof, $X = h^{-1}[Y] \in \mathcal{J}_{SZ}$. Hence $Y = h[X] \in h\{\mathcal{J}_{SZ}\}$. ■

As we mentioned at the beginning of this paper, the concept of Sierpiński-Zygmund sets is a generalization of the concept of Sierpiński-Zygmund functions. One of the questions related to the family SZ of Sierpiński-Zygmund functions is for how "big" families $F \subseteq \mathbb{R}^{\mathbb{R}}$ we can find a function $g \in \mathbb{R}^{\mathbb{R}}$ such that $g + F \subseteq SZ$. (See e.g. [3].) Similar question can be asked in the case of Sierpiński-Zygmund sets. This leads to the following definition.

Definition 7 A set $X \subseteq \mathbb{R}^2$ is called *SZ-shiftable*, if there exists a function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f + X$ is SZ-set.

We denote the family of all SZ-shiftable sets by SZ_{shift} . Obviously $\mathcal{J}_{SZ} \subseteq SZ_{shift}$, so SZ_{shift} is not empty.

Lemma 8 Let $X \subseteq \mathbb{R}^2$. If for all $x \in \mathbb{R}$ and $A \subseteq [\mathbb{R}]^{<c}$ there exists an $a \in \mathbb{R}$ such that $(a + A) \cap X_x = \emptyset$, then A is SZ-shiftable.

PROOF. Let $x: \omega < c$ and $f: \omega < c$ be the sequences of all real numbers and all continuous functions defined on a G -subset of \mathbb{R} , respectively. We will define a function $f: \mathbb{R} \rightarrow \mathbb{R}$ which shifts X into \mathcal{J}_{SZ} , using transfinite induction. For every $\alpha < c$ we choose $f(x_\alpha) \in \mathbb{R}$ with the property that

1. $f(x_\alpha) + X_x = \emptyset$ for all $x \in \mathbb{R}$.

It may also be of interest to determine whether SZ_{shift} is closed under the union operation. Fact 2 states, in particular, that the union of two SZ-sets is also an SZ-set. Thus, the natural question that appears here is whether the same is true for SZ-shiftable sets. It turns out not to be the case.

Example 10 *There exist $A_1, A_2 \in SZ_{shift}$ such that $A_1 \cup A_2 = \mathbb{R}^2 \notin SZ_{shift}$.*

PROOF. Put A_1 to be the set X from Lemma 3 and A_2 to be its complement. Based on Lemma 8 A_2 is SZ-shiftable. Next, notice that $A_1 \in \mathcal{J}_{SZ} \subseteq SZ_{shift}$. Finally, $A_1 \cup A_2 = \mathbb{R}^2$ and obviously \mathbb{R}^2 is not in SZ_{shift} . ■

Before we finish let us make a comment about [8, Theorem 2 (1)] which says: MA implies that for every finite family F of real functions there exists an almost continuous function g (each open subset of \mathbb{R}^2 containing the graph of g contains also the graph of a continuous function) such that $g + f$ is Sierpiński-Zygmund for every $f \in F$. Note that this result can be expressed using the notion of SZ-sets. Under MA the following holds:

If, for some fixed $n \in \mathbb{N}$, every vertical section of the set $X \subseteq \mathbb{R}^2$ has at most n elements then there exists an almost continuous function $f: \mathbb{R} \rightarrow \mathbb{R}$ such that $f + X \in \mathcal{J}_{SZ}$.

We generalize the above result.

Theorem 11

Lemma 13 (MA) Let $F = F_A$ be a family such that $|F| = c$. There exists a $g \in \text{SZ}(A)$ such that $g + F \in \text{SZ}(A)$ and for every blocking set $B \subseteq \mathbb{R}^2$ there is a non-empty open interval $I_B \subseteq \text{dom}(B)$ with the property that $\text{dom}(B \cap g)$ is dense in I_B .

Lemma 14 [8, Lemma 13] (MA) Let $\{f_i\}_1^n \subseteq \mathbb{R}^{\mathbb{R}}$, $n = 1, 2, \dots$. There exists $\{f_i\}_1^n \subseteq F_A$ such that $f_i|_{A_i} \in C_{<c}(A_i)$, where $A_i = \{x : f_i = f_j\}$.

Note that Lemmas 13 and 14 imply the following.

() (MA) Assume that $F \subseteq \mathbb{R}^{\mathbb{R}}$ is finite and $A \subseteq \mathbb{R}$ is everywhere of second category. Then there exists a function $g: A \subseteq \mathbb{R}$ such that $g + F \in \text{SZ}(A)$ and $\text{dom}(g \cap B)$ is dense in some non-empty open interval I_B for every blocking set B .

PROOF. Let us consider the partition $\{H_n : n \in \mathbb{N}\}$ of \mathbb{R} , where H_n is defined by $H_n = \{x \in \mathbb{R} : |X_x| = n\}$. Let $G_n \subseteq \mathbb{R}$ be a maximal open set such that H_n is everywhere of second category in G_n . Such a set can be easily constructed. Simply define G_n as the interior of the set $\mathbb{R} \setminus \bigcup_{I \in I_n} I$, where I_n is the set of all open intervals in which H_n is meager.

We claim that for every $n < \omega$, there exists a function $g_n: (G_n \cap H_n) \subseteq \mathbb{R}$ such that $g_n + X = \{x, g_n(x) + y : x \in (G_n \cap H_n), x, y \in X\} \in \mathcal{J}_{\text{SZ}}$ and $\bigcup_{n < \omega} g_n$ intersects every blocking set B .

First observe that this claim implies the conclusion of the theorem. Put $g: \mathbb{R} \subseteq \mathbb{R}$ to be an extension of $\bigcup_{n < \omega} g_n$ such that $[g/(\mathbb{R} \setminus \bigcup_{n < \omega} G_n \cap H_n)] + X$ is an SZ-set. This extension exists based on Corollary 9. Thus, $g + X$ is the union of countable many SZ-sets. Consequently, $g + X \in \mathcal{J}_{\text{SZ}}$. Clearly, g intersects every blocking set, so g is almost continuous.

To complete the proof we need to show the above claim. Fix an $n < \omega$ and put $A_n = (G_n \cap H_n) \cup \bigcup_{I \in I_n} I$. The set A_n is everywhere of second category. Notice also that the part of X contained in $(G_n \cap H_n) \times \mathbb{R}$ can be covered by n functions f_1, \dots, f_n from \mathbb{R} to \mathbb{R} . So, by (), there exists a function $g_n: A_n \subseteq \mathbb{R}$ such that $g_n + \{f_1, \dots, f_n\} \in \text{SZ}(A_n)$ and $\text{dom}(g_n \cap B)$ is dense in some non-empty open interval I_B for every blocking set B . Thus, if we define $g_n = g_n|(G_n \cap H_n)$ then $g_n + X \in \mathcal{J}_{\text{SZ}}$.

What remains to prove is that $\bigcup_{n < \omega} g_n$ intersects every blocking set B . Notice that $I_B \cap G_n = \emptyset$ for some n . Thus, $g_n \cap B = \emptyset$. Consequently, $B \cap \bigcup_{n < \omega} g_n \cap B = g$. This finishes the proof. ■

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