functions are the only continuous solutions, was first proved by A. L. Cauchy [1

Proof. (i) Let f LIF(R^n) and g AD(R^n). Fix $x_1, \ldots, x_k$ R^n and $q_1, \ldots q_k$
Q

This choice is possible since

$$\left|\bigcup_{f\in F} ((h+f)[\{x_\xi : \xi < \alpha\}] \cup \{f(x_\alpha)\})\right| \le (\alpha+1)|F| < \mathfrak{c}.$$

It is easy to see that all the required properties of h are preserved. This ends the proof of A(LIF(R

Notice here that A(LIF) = $\mathfrak{c}$ (Fact 2.3 (v)) implies, in particular, that every function from $\mathbb{R}^{\mathbb{R}}$ can be written as the algebraic sum of two linearly independent functions. In other words LIF + LIF = $\mathbb{R}^{\mathbb{R}}$. Since we only found the upper bound for A(HF), it wo16Td[(can)-347(b)-27spper bo12313(A)-350((r)v(e)-350d[(y50((r))1(d))-1(.er)-470(tn)1(dg50((r

Proof. Notice first that if $|LC(f, 2)| = \mathfrak{c}$ then case (a) holds with $Z = \{0$

From (•) we see that if $\mathrm{Lin}_{\mathbb{Q}}(x_1^\xi, x_2^\xi, x_3^\xi) \cap \mathrm{Lin}_{\mathbb{Q}}(X) = \{0\}$ holds for $\mathfrak{c}$-many ξ then the set Z satisfies the condition $|\ \ _{z \in Z} \mathrm{LC}(f, 2, z)| = \mathfrak{c}$. Obviously $Z \quad [\mathbb{R}^n]^{<\mathfrak{c}}$. Thus, case (a) holds.

Summarizing the above discussion, we just need to consider a situation when $\dim(\{x_1^\xi, x_2^\xi, x_3^\xi\}) = 2$ and $\mathrm{Lin}_{\mathbb{Q}}(x_1^\xi, x_2^\xi, x_3^\xi) \cap \mathrm{Lin}_{\mathbb{Q}}(X) = \{0\}$ for all ξ. Recall that $q_1 x_1^\xi + q_2 x_2^\xi + q_3 x_3^\xi = 0$, where $q_1, q_2, q_3 \quad \mathbb{Q} \setminus \{0\}$. If two of $x_1^\xi, x_2^\xi, x_3^\xi$ were dependent over $\mathbb{Q}$ then we would have $\dim(\{x_1^\xi, x_2^\xi, x_3^\xi\}) \le 1$. Thus, $x_1^\xi, x_2^\xi, x_3^\xi$ are pairwise independent. Now it is easy to see that case (b) holds. ■

Lemma 3.8. *Let $X \quad [\mathbb{R}^n]^{<\mathfrak{c}}$, x / X, and $y \quad \mathbb{R}$. Suppose also that $h, g\colon X \to \mathbb{R}$ are functions linearly independent over $\mathbb{Q}$. Then there exist extensions h', g' of h and g onto $X \quad \{x\}$ such that h' and g' are linearly independent over $\mathbb{Q}$ and $h'(x) + g'(x) = y$.*

Proof. Choose $h'(x) \quad \mathbb{R} \setminus \mathrm{Lin}_{\mathbb{Q}}(h[X] \quad g[X] \quad \{y\})$. This choice is possible since $|\mathrm{Lin}_{\mathbb{Q}}(h[X] \quad g[X] \quad \{y\})| < \mathfrak{c}$. Then define $g'(x) = y - h'(x)$. It is easy to d[(x)]TF89.963TfmF(x)]T3Tf-301.463

holds because $f(-a_0) + f(a_0) = c$ and $m_0 = c$ if $c = 0$. Thus $0, c$, $0, m_0$ $\mathrm{Lin}_{\mathbb{Q}}(h|A_0) \cap \mathrm{Lin}_{\mathbb{Q}}(g|A_0)$. It is easily seen that $h|A_0$ and $g|A_0$ satisfy (a) and (b).

x_ξ $\mathrm{dom}(h_\xi) = \mathrm{dom}(g_\xi)$ and v_ξ $\mathrm{Lin}_{\mathbb{Q}}(h_\xi) \cap \mathrm{Lin}_{\mathbb{Q}}(g_\xi)$, where h_ξ and g_ξ denote the extensions obtained in the step ξ.

Let $\beta < \mathfrak{c}$. Assume that v_β / $\mathrm{Lin}_{\mathbb{Q}}(\square_{\xi<\beta} h_\xi)$ $\mathrm{Lin}_{\mathbb{Q}}(\square_{\xi<\beta} g_\xi)$. Choose an a $\mathbb{R} \setminus \mathrm{Lin}_{\mathbb{Q}}(\mathrm{dom}(\square_{\xi<\beta} h_\xi))$ and define $h(x)$ by $0, h(x) = \frac{1}{2} v_\beta$ for x $\{-a, a\}$. Put also $g(x) = f(x) - h(x)$. Since $f(-a) + f(a)$ $\mathrm{LC}(f)$, (3.3) implies that v_β $\mathrm{Lin}_{\mathbb{Q}}($x

The inductive construction of functions h and g is somewhat similar to the one from the previous case. So assume that $\beta < \mathfrak{c}$ and the construction has been carried out for all $\xi < \beta$. If $v_\beta \notin \mathrm{Lin}_{\mathbb{Q}}(h)$ then let $X = \mathrm{dom}(h) = \mathrm{dom}(g)$ and $Y \in [\mathbb{R}]^{<\mathfrak{c}}$ be such a set that $\mathrm{Lin}_{\mathbb{Q}}(g \cup \{v_\beta\}) \subseteq \mathbb{R}^n \times Y$. By Property 2 (b), there exist $p_1, p_2, p_3 \in \mathbb{Q} \setminus \{0\}$ and pairwise independent $x_1, x_2, x_3 \in \mathbb{R}^n$ such that $\sum_1^3 p_i x_i = 0$, $\mathrm{Lin}_{\mathbb{Q}}(x_1, x_2, x_3) \cap \mathrm{Lin}_{\mathbb{Q}}(X) = \{0\}$, and $\sum_1^3 p_i f(x_i) \notin Y$.

We extend h and g onto $\{x_1, x_2, x_3\}$. Choose $h(x_1), h(x_2), h(x_3) \in \mathbb{R}$ in such a way that

$$\sum_1^3 p_i \langle x_i, h(x_i)\rangle = \left\langle 0, \sum_1^3 p_i h(x_i)\right\rangle = v_\beta.$$

Then put $g(x_i) = f(x_i) - h(x_i)$ for $i \le 3$. Obviously $v_\beta \in \mathrm{Lin}_{\mathbb{Q}}(h)$ and